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Tidal rectification in lateral viscous boundary layers of a semi-enclosed basin

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The rectified flow, induced by divergence of the vorticity flux in lateral oscillatory viscous boundary layers along the sidewalls of a semi-enclosed basin, is studied as a function of the Strouhal number, κ, equivalent to the Reynolds number of the viscous inner oscillatory boundary layer, and of the Stokes number. The squared ratio of these numbers defines another Reynolds number, measuring the strength of the self-advection by the residual flow. For strong self-advection the residual current decays to zero in an outer boundary, its width being large compared to the width of the inner layer. The regimes of small, moderate and strong self-advection are analysed.

1. Introduction

In many shallow seas, where the tidal current amplitude is larger than about 0.5 ms^{-1}, it is known that if the local velocity field is averaged over one or more tidal cycles the result is unequal to zero. Many mechanisms may generate such a constant flow (Huthnance 1981; Zimmerman 1981). Here we will study the effect of lateral frictional boundary layers. As shown by Yasuda (1980), in a semi-enclosed tidal basin they induce a rectified mean circulation with an intense outward flow along the boundary and a weak inward flow in the central region. The mechanism, essentially, can be understood as a divergence of the tidal averaged flux of vorticity, produced by viscous friction along the sidewalls, ultimately balanced by viscous vorticity diffusion (Zimmerman 1981). The crucial parameter, on which the strength of the retification depends, is the Reynolds number based on the ratio of longitudinal vorticity advection and lateral vorticity diffusion.

Let U be a velocity scale, L the length of the semi-enclosed basin, ν the horizontal (turbulent) vorticity and σ the basic frequency of the tidal flow. Then the lateral boundary-layer thickness due to oscillatory flow along the sidewalls is

$$\delta = (\nu/\sigma)^{\frac{1}{2}}. \tag{1.1}$$

Hence we can define a Reynolds number as

$$Re = \frac{U^2/L}{\nu U/\delta^2} = \frac{U\delta^2}{\nu L} = \frac{U}{\sigma L} = \kappa. \tag{1.2}$$

Thus for the dynamics concerned, the Reynolds number is equivalent to the ratio of the tidal excursion U/σ and the basin length L, being the Strouhal number κ (Zimmerman 1981).

Yasuda (1980) studied the rectified flow for small κ. He considered the case of a small Stokes number $E = \delta/B$ (B being the half-width of the basin), when the oscillatory flow exhibits a boundary-layer character with the vorticity being constrained to a small area near the sidewalls. Yet the rectified current extended over the whole basin width, having a finite velocity at the mid-axis of the basin which is independent of the Stokes number.

However, from a more detailed analysis it appears that this behaviour is only to be expected as long as the divergence of the overall vorticity flux (measured by a Strouhal number $\kappa \ll 1$) is small or of comparable order with respect to vorticity diffusion (measured by E^2). This is clearly demonstrated by Riley (1967) for the general case of oscillatory viscous flow. He also points out that other situations are more complicated and must be studied with different techniques.

In the case where diffusion is small, even compared to advection, the theory of double boundary layers can be used, see Riley (1965, 1967), Stuart (1963, 1966) and Grotberg (1984). Originally the theory was particularly developed for the archetypal example of an oscillating cylinder in a viscous fluid (Schlichting 1932). One of its main points is that the rectified flow also exhibits a boundary-layer character, its width being much larger than the width of the Stokes layer. Inside this 'outer layer' the rectified current can decay to zero. Physically, the existence of the outer layer is related to self-advection of residual vorticity by the residual current, a process that has been neglected in Yasuda's (1980) approach. The strength of self-advection, relative to viscous dissipation, is measured by a Reynolds number based on the residual velocity scale, of order κU, the basin lengthscale L and the viscosity, i.e.

$$R_* = \frac{\kappa U L}{v} = \kappa^2 E^{-2}. \tag{1.3}$$

The regime studied by Yasuda (1980) applies to $R_* \ll 1$ in which self-advection by the residual flow can be neglected. Here, we extend his analysis to the regime $R_* \geqslant O(1)$.

We start from the shallow-water equations for a homogeneous fluid with corresponding boundary conditions, properly scaled in §2. In passing we note that after scaling it appears that Yasuda's (1980) solution is incomplete in the sense that an additional rectification mechanism, of the same strength as the one discussed by him, cannot be neglected, namely lateral vorticity advection. We therefore recalculate the rectified current velocity field for the regime $R_* \ll 1$ in §3. The case $R_* \geqslant O(1)$ is considered in §4. Finally in §5 a discussion of the results and some conclusions are presented.

2. Scaling of the basic equations

We consider a semi-enclosed basin having a uniform equilibrium depth, H, a length L and a width $2B = 2L$. This is done to keep the subsequent analysis as simple as possible. Later on the results will be generalized to $B/L = O(1)$.

The shallow-water equations of motion for a rotating homogeneous fluid are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \zeta}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right], \tag{2.1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu = -g\frac{\partial \zeta}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]. \tag{2.2}$$

The continuity equation is given by

$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}[(H+\zeta)\,u]+\frac{\partial}{\partial y}[(H+\zeta)\,v] = 0. \tag{2.3}$$

Here u and v are the horizontal components of the velocity vector, assumed to be vertically uniform as we have left out vertical turbulent momentum transfer. From Yasuda (1980) it is clear that the inclusion of the latter process does not add anything of importance to the dynamics we are concerned with here, which is mainly the generation of the vertical vorticity component by sidewall friction, represented by the (turbulent) viscosity coefficient v in the right-hand side of (2.1)–(2.2). Furthermore f is the Coriolis parameter, g the acceleration due to gravity and ζ the height of the surface of the fluid above the reference level.

In looking for the dimensionles form of (2.1)–(2.3) we scale x and y by L, t by σ^{-1} (σ the tidal frequency), u and v with a velocity scale U (the velocity amplitude in the middle of the open boundary, say), whereas the continuity equation suggests scaling ζ with $UH/\sigma L$. We *a priori* assume that the ratio σ/f is of the order 1. Defining the following non-dimensional parameters;

$$\left.\begin{aligned}
&\kappa \text{ (Strouhal number)} = \frac{U}{\sigma L},\\
&F \text{ (Froude number)} = \frac{U}{(gH)^{\frac{1}{2}}},\\
&\lambda = \frac{2\pi \text{ basin length}}{\text{wavelength}} = \frac{\sigma L}{(gH)^{\frac{1}{2}}} = F\kappa^{-1},\\
&E = \frac{\text{viscous boundary layer width}}{\text{halfbasin width}} = \frac{\delta}{L}
\end{aligned}\right\} \tag{2.4}$$

the equations of motion and the continuity equation are

$$\frac{\partial u}{\partial t}+\kappa\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]-\frac{f}{\sigma}v = -\frac{1}{\lambda^2}\frac{\partial \zeta}{\partial x}+E^2\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right], \tag{2.5}$$

$$\frac{\partial v}{\partial t}+\kappa\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]+\frac{f}{\sigma}u = -\frac{1}{\lambda^2}\frac{\partial \zeta}{\partial y}+E^2\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right], \tag{2.6}$$

$$\frac{\partial \zeta}{\partial t}+\kappa\left[u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}\right] = -(1+\kappa\zeta)\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]. \tag{2.7}$$

All variables have to be understood as being dimensionless and scaled according to the scheme given above. We search for non-transient solutions, satisfying the boundary conditions

$$\left.\begin{aligned}
&u = \sin t \quad \text{at } x = 0,\\
&u = 0 \quad \text{at } x = 1,\quad y = 0 \quad \text{and } y = 2,\\
&v = 0 \quad \text{at } x = 0,\quad x = 1,\quad y = 0 \quad \text{and } y = 2.
\end{aligned}\right\} \tag{2.8}$$

From here on we shall assume that

$$E \ll 1, \tag{2.9}$$

giving rise to a singular perturbation problem as E multiplies the highest-order derivatives in (2.5)–(2.7). Furthermore we consider basins having a characteristic lengthscale which is much smaller than the tidal wavelength. From (2.4) it then follows that

$$\lambda \ll 1, \tag{2.10}$$

and as a consequence rotation effects will not be of importance.

After substitution of the regular expansions

$$\left.\begin{aligned} u(x,y,t) &= U_0(x,y,t)+EU_1(x,y,t)+\dots,\\ v(x,y,t) &= V_0(x,y,t)+EV_1(x,y,t)+\dots,\\ \zeta(x,y,t) &= Z_0(x,y,t)+EZ_1(x,y,t)+\dots, \end{aligned}\right\} \tag{2.11}$$

it follows that the zeroth-order momentum equations can be linearized for any Strouhal number κ, since

$$\kappa\lambda^2 = \lambda F \ll 1, \tag{2.12}$$

the estimate following from (2.10) and the assumption that the Froude number is small in order to prevent breaking tidal waves.

Evidently the zeroth-order equations in the regular expansion read

$$\left.\begin{aligned} &\frac{\partial Z_0}{\partial x} = 0,\\ &\frac{\partial Z_0}{\partial y} = 0,\\ &\frac{\partial Z_0}{\partial t} = -(1+\kappa Z_0)\left[\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right]. \end{aligned}\right\} \tag{2.13}$$

Note that we write capitals for the regular expansions. The solution of (2.13), subject to the slip boundary conditions

$$\left.\begin{aligned} U_0 &= \sin t \quad \text{at } x=0,\\ U_0 &= 0 \quad \text{at } x=1,\\ V_0 &= 0 \quad \text{at } y=0 \text{ and } y=2, \end{aligned}\right\} \tag{2.14}$$

reads

$$\left.\begin{aligned} &U_0 = (1-x)\sin t, \quad V_0 = 0,\\ &Z_0 = \frac{-1+\exp(-\kappa\cos t)}{\kappa}. \end{aligned}\right\} \tag{2.15}$$

This is the well-known expression for a standing shallow-water gravity wave, which is valid under the conditions (2.9) and (2.10). In the same way the first-order regular system can be solved. For the lateral velocity component we obtain

$$V_1 = 0, \tag{2.16}$$

which will be used later on.

As the solution (2.14) does not include the viscous sidewall layers, necessary to bring the tangential velocity components to zero along the walls, we have to correct the velocity field near the sidewalls by introducing boundary layers. Let

$$y' = \frac{L}{\delta}y \tag{2.17}$$

be a lateral stretched coordinate near the boundary $y = 0$, based on the already dimensionless coordinate y (scaled by L), and assume that we have to rescale the lateral velocity component by $\delta U/L$, as suggested by mass balance in the viscous boundary layer. Then we have the expansions

$$\left.\begin{aligned} u &= U_0(x,t)+u_0(x,y',t)+E\{U_1(x,y,t)+u_1(x,y',t)\}+\dots,\\ v &= Ev' = Ev_1(x,y',t)+\dots,\\ \zeta &= Z_0(t)+\zeta_0(y',t)+E\{Z_1(x,y,t)+\zeta_1(x,y',t)\}+\dots, \end{aligned}\right\} \tag{2.18}$$

where capitals refer to the variables in the regular expansion and lower-case letters to the boundary-layer corrections. These series should be substituted in the rescaled equations of motion:

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+\kappa\left[u\frac{\partial u}{\partial x}+v'\frac{\partial u}{\partial y'}\right]-E\frac{f}{\sigma}v' &= -\frac{1}{\lambda^2}\frac{\partial \zeta}{\partial x}+E^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y'^2},\\ E^2\left\{\frac{\partial v'}{\partial t}+\kappa\left[u\frac{\partial v'}{\partial x}+v'\frac{\partial v'}{\partial y'}\right]\right\}+E\frac{f}{\sigma}u &= -\frac{1}{\lambda^2}\frac{\partial \zeta}{\partial y'}+E^2\left\{E^2\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y'^2}\right\},\\ \frac{\partial \zeta}{\partial t}+\kappa\left[u\frac{\partial \zeta}{\partial x}+v'\frac{\partial \zeta}{\partial y'}\right] &= -(1+\kappa\zeta)\left[\frac{\partial u}{\partial x}+\frac{\partial v'}{\partial y'}\right]. \end{aligned}\right\} \tag{2.19}$$

To zeroth order in E we find, after some manipulations and use of the zeroth-order regular equations (2.13), that

$$\frac{\partial u_0}{\partial x}+\frac{\partial v_1}{\partial y'} = 0, \tag{2.20}$$

showing that to this approximation the corrective velocity field is free of divergence, hence $\zeta_0(y',t) = 0$. Thus a streamfunction ψ may be introduced, such that

$$u_0 = -\frac{\partial \psi'}{\partial y'},\quad v_1 = \frac{\partial \psi'}{\partial x}, \tag{2.21}$$

the prime showing that the dimensionless stream function ψ' is scaled by the tidal velocity amplitude and the Stokes boundary-layer width δ. The dynamics will now be governed by a vorticity equation.

First we note that the inviscid regular field (2.15) is free of rotation. Obviously vorticity arises only by the presence of frictional boundary layers. Its dimensionless form (scaled by U/δ) reads

$$\omega = E^2\frac{\partial v'}{\partial x}-\frac{\partial u}{\partial y'}. \tag{2.22}$$

From the rescaled momentum equations a vorticity equation can be derived. Substitution of the expansions (2.18) gives in zeroth order

$$\frac{\partial \omega_0}{\partial t}+\kappa[U_0+u_0]\frac{\partial \omega_0}{\partial x}+\kappa v_1\frac{\partial \omega_0}{\partial y'}+\kappa\omega_0\frac{\partial U_0}{\partial x} = \frac{\partial^2 \omega_0}{\partial y'^2}, \tag{2.23}$$

using (2.16) and (2.20). Writing ω_0 in terms of the stream function by means of (2.21) and (2.22) and substituting (2.15) for U_0, we finally obtain

$$\frac{\partial}{\partial t}\frac{\partial^2 \psi'}{\partial y'^2}+\kappa(1-x)\sin t\frac{\partial}{\partial x}\frac{\partial^2 \psi'}{\partial y'^2}+\kappa J'\left(\psi',\frac{\partial^2 \psi'}{\partial y'^2}\right)-\kappa\sin t\frac{\partial^2 \psi'}{\partial y'^2} = \frac{\partial^4 \psi'}{\partial y'^4}. \tag{2.24}$$

The Jacobian J' has its usual meaning. It describes the advection of vorticity by the boundary-layer flow and is the cause of the principal nonlinearity of the vorticity equation. In terms of the rescaled variables

$$y = Ey', \quad \psi = E\psi', \tag{2.25}$$

(2.24) reads

$$\frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial y^2}+\kappa(1-x)\sin t\frac{\partial}{\partial x}\frac{\partial^2\psi}{\partial y^2}+\kappa J\left(\psi,\frac{\partial^2\psi}{\partial y^2}\right)-\kappa\sin t\frac{\partial^2\psi}{\partial y^2} = E^2\frac{\partial^4\psi}{\partial y^4}. \tag{2.26}$$

In order to have the problem of solving (2.26) fully posed we finally introduce the following five boundary conditions:

$$\left.\begin{aligned} &\psi = 0 \quad \text{at } x = 1, \quad y = 0, \quad y = 1, \\ &\frac{\partial\psi}{\partial y} = (1-x)\sin t \quad \text{at } y = 0, \\ &\frac{\partial^2\psi}{\partial y^2} = 0 \quad \text{at } y = 1. \end{aligned}\right\} \tag{2.27}$$

The first and second are obvious choices. The fourth one is in fact the no-slip condition as the regular velocity U_0 and the boundary-layer correction together must vanish at the sidewall. The third and fifth condition naturally arise from the symmetry of the flow about the mid-basin axis at $y = 1$. Note that the solution to (2.26)–(2.27) is valid in the lower half of the basin. Once it is known the solution in the upper half of the basin can be obtained by reflection in $y = 1$. Equation (2.26) describes the generation of vorticity by the no-slip conditions (2.27) at the sidewalls. Since the latter are periodic in time, the resulting stream function will also have a periodic character, but, as can be seen in (2.26), all terms proportional to κ may produce higher harmonics as well as a rectified time-independent component. Note that we do not require the longitudinal component of the residual velocity to vanish at $y = 1$. We expect that for sufficiently wide basins the dynamics of the flow itself leads to vanishing residual velocity at the mid-basin axis.

Evidently solutions of (2.26)–(2.27) are controlled by the non-dimensional parameters κ and E, which are the Strouhal number and Stokes number, respectively. In view of the discussion in the introduction, it is their squared ratio, $R_* = \kappa^2 E^{-2}$ that determines whether or not the induced rectified flow has boundary-layer character. For $R_* \ll 1$ it has not and therefore it may show a finite value at $y = 1$ for $E \to 0$. On the other hand, for $R_* \gg 1$ another boundary layer develops in which the residual velocity tends to zero far away from the lateral boundaries. We shall now analyse these two regimes separately. Throughout these analyses we assume $\kappa \ll 1$.

3. Residual circulation for small Strouhal numbers and weak self-advection

In this case we have $\kappa \ll 1$ and $\kappa \ll E$. From the results of Riley (1967) it then follows that an approximate solution of (2.26) can be written as

$$\psi = \psi_0 + \kappa\psi_1 + \dots. \tag{3.1}$$

where ψ_0 and ψ_1 depend explicitly on E. To zeroth order we have

$$\frac{\partial}{\partial t}\frac{\partial^2\psi_0}{\partial y^2} - E^2\frac{\partial^4\psi_0}{\partial y^4} = 0, \tag{3.2}$$

subject to the same boundary conditions for ψ_0 as for ψ in (2.27). The solution is straightforward as in fact (3.2) describes a non-dimensional diffusion of vorticity with periodic boundary conditions. We find

$$\psi_0(x,y,t) = (1-x)\,\phi_0(y,t), \tag{3.3}$$

where

$$\left.\begin{aligned} \phi_0 &= \hat{\phi}_0\,\mathrm{e}^{\mathrm{i}t}+\hat{\phi}_0^*\,\mathrm{e}^{-\mathrm{i}t}, \\ \hat{\phi}_0 &= \frac{-\sinh\,[\mathrm{i}^{\frac{1}{2}}E^{-1}(1-y)]+(1-y)\sinh\,(\mathrm{i}^{\frac{1}{2}}E^{-1})}{2\mathrm{i}\{\mathrm{i}^{\frac{1}{2}}\cosh\,(\mathrm{i}^{\frac{1}{2}}E^{-1})-\sinh\,(\mathrm{i}^{\frac{1}{2}}E^{-1})\}}, \end{aligned}\right\} \tag{3.4}$$

with the asterisk denoting a complex conjugation.

Rectification now arises to first order in κ. To this order the stream function obeys

$$\frac{\partial}{\partial t}\frac{\partial^2\psi_1}{\partial y^2}-E^2\frac{\partial^4\psi_1}{\partial y^4} = -\left\{(1-x)\sin t\frac{\partial}{\partial x}\frac{\partial^2\psi_0}{\partial y^2}+J\left(\psi_0,\frac{\partial^2\psi_0}{\partial y^2}\right)-\sin t\frac{\partial^2\psi_0}{\partial y^2}\right\}, \tag{3.5}$$

subject to the same boundary conditions (2.27) for ψ, except the fourth one which should be replaced by

$$\frac{\partial\psi_1}{\partial y} = 0 \quad \text{at } y = 0, \tag{3.6}$$

as ψ_0 already satisfies the fourth boundary condition of (2.27).

The time-independent, rectified, part of the solution of (3.5) can be obtained by applying a time-averaging operator to (3.5). Let this operator be denoted by a bar:

$$\overline{(\cdot)} = \frac{1}{2\pi}\int_0^{2\pi}(\cdot)\,\mathrm{d}t. \tag{3.7}$$

Then (3.5) reads

$$E^2\frac{\partial^4\overline{\psi}_1}{\partial y^4} = (1-x)\,\overline{\sin t\frac{\partial}{\partial x}\frac{\partial^2\psi_0}{\partial y^2}}+\overline{J\left(\psi_0,\frac{\partial^2\psi_0}{\partial y^2}\right)}-\overline{\sin t\frac{\partial^2\psi_0}{\partial y^2}}. \tag{3.8}$$

Substituting (3.3) and (3.4) into (3.8), performing the time averaging, integrating over y and using the boundary conditions, we find a rather complicated solution of (3.8), involving hyperbolic and trigonometric functions of y. The solution is presented in Appendix A. Formally (3.8) can be solved for any value of E. However, only the limit $E\rightarrow 0$ is physically significant, see (2.9). The residual stream function $\overline{\psi}$ and the components $\overline{u}$ and $\overline{v}$ of the residual current are obtained by expanding the results of Appendix A in powers of $a^{-1} = \sqrt{2E}$. It then follows

$$\begin{aligned} \overline{\psi} = \kappa(1-x)\Big\{&-\frac{1}{4a}\mathrm{e}^{-ay}[\tfrac{1}{2}\mathrm{e}^{-ay}+2[\sin(ay)+\cos(ay)] \\ &-(1-y)[\sin(ay)-\cos(ay)]]+\tfrac{3}{8}(1-y)^3-\tfrac{3}{8}(1-y)+\frac{7}{8a}\Big\}+O\left(\frac{\kappa}{a}\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \overline{u} = -\frac{\partial\overline{\psi}}{\partial y} = \kappa(1-x)\{&-\mathrm{e}^{-ay}[\tfrac{1}{4}\mathrm{e}^{-ay}+\sin(ay)+\tfrac{1}{2}(1-y)\cos(ay)] \\ &+\tfrac{9}{8}(1-y)^2-\tfrac{3}{8}\}+O\left(\frac{\kappa}{a}\right), \end{aligned} \tag{3.10}$$

$$\overline{v} = \frac{\partial\overline{\psi}}{\partial x}. \tag{3.11}$$

Here we have included some of the $O(\kappa/a)$ terms in $\overline{\psi}$ which contribute, after differentiation, to the $O(\kappa)$ approximation of $\overline{u}$. Furthermore, we have chosen the

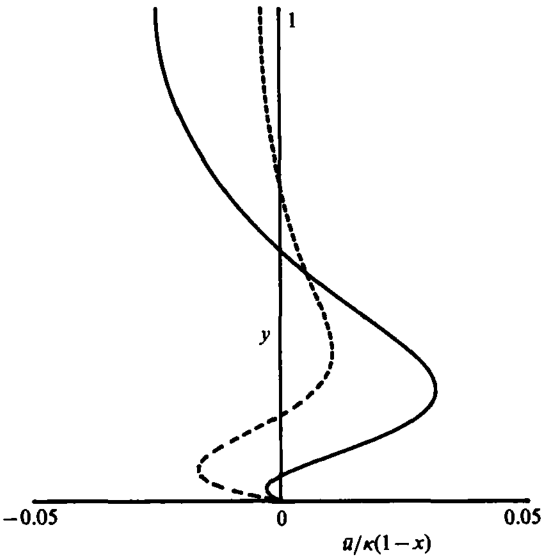

FIGURE 1. Lateral profiles of ——, the non-dimensional longitudinal residual current solution (3.10), and ---, the Yasuda solution; $E^{-1} = 15$. Flow in the central region is positive.

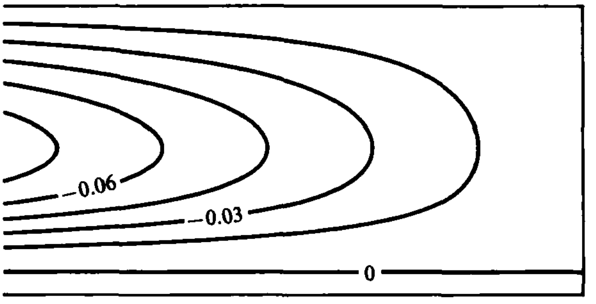

FIGURE 2. Stream function pattern of the residual current solution (3.10).

expansions such that $\bar{\psi}$ obeys all boundary conditions. Note that $\bar{u}$, apart from an exponential decay in the stretched coordinate $y' = y/E$, exhibits a parabolic profile in the unstretched coordinate y. That means that for $E \to 0$ (such that $\kappa^2 E^{-2}$ remains small) the residual velocity at the mid-basin axis keeps a finite value:

$$\bar{u}(y = 1) = -\tfrac{3}{8}\kappa(1-x) + O(\kappa E). \tag{3.12}$$

Obviously the intensity of the residual current is proportional to the Strouhal number as long as κ and R_* are small, and for that matter the residual current velocity in dimensional form is proportional to the square of the undisturbed tidal velocity amplitude, a result already derived by Yasuda (1980). However there is a qualitative disagreement between our solution and Yasuda's, in that an additional term is present in (3.10). This can be traced back to the basic equation used here, namely equation (2.24), and to the one used by Yasuda (1980). It appears that the latter author only takes the longitudinal advection of vorticity into account. However, as our scaling shows, lateral vorticity advection in the boundary layer is of the same order as the former term, and thus has to be taken into account as well.

In figure 1 the lateral profiles of both the solution (3.10) and the Yasuda solution are shown for $E^{-1} = 15$ (a characteristic value for tidal basins, as will be discussed in §5). They have the same qualitative behaviour, namely an outward flux near the sidewall, an inward flux just outside the boundary layer and again an outflux in the central region. The difference between them is entirely due to the lateral advection term. Including this contribution means that vorticity is advected laterally over the bay, obviously resulting in a weaker outflux in the boundary layer and stronger fluxes in the central region. In figure 2 a contourplot of the associated residual streamfunction is shown. Of course, in order for the profiles of figure 1 and figure 2 to be meaningful, R_* should be small; i.e. for $E^{-1} = 15, \kappa \ll \frac{1}{15}$. This assumption is, however, not realistic to tidal basins where $\kappa \sim 0.7$, see §5. Therefore in the next section we will study the residual current dynamics for R_* of order 1 and larger.

4. Effect of self-interaction on the residual circulation for small Strouhal numbers

In the case of moderate to large values of R_* there is an 'outer region', outside the Stokes layer near the wall, where self-advection due to the residual current and lateral diffusion of residual vorticity are of comparable importance. We then have to match the profile in the Stokes layer to the profile in the outer region in such a way that in summation these profiles obey the boundary conditions.

Using now explicitly the boundary-layer character of the inner layer, its dynamics is given by (2.24). Expanding ψ' in κ and E^{-1} we have to zeroth order

$$\frac{\partial}{\partial t}\frac{\partial^2 \psi'_0}{\partial y'^2} = \frac{\partial^4 \psi'_0}{\partial y'^4}, \tag{4.1}$$

subject to

$$\left.\begin{aligned} &\psi'_0 = 0, \quad \frac{\partial \psi'_0}{\partial y'} = (1-x)\sin t \quad (y' = 0), \\ &\psi'_0 = 0, \quad \frac{\partial^2 \psi'_0}{\partial y'^2} \rightarrow 0 \quad (y' \rightarrow \infty), \end{aligned}\right\} \tag{4.2}$$

$$\psi'_0 = 0 \quad (x = 1),$$

where the boundary conditions arise from the fact that an oscillatory outer solution at this order vanishes, as is obvious from (2.26), expanding in κ and E^{-1}. The solution of (4.1) reads:

$$\left.\begin{aligned} \psi'_0 &= (1-x)\{\hat{\psi}'_0 \mathrm{e}^{\mathrm{i}t} + \hat{\psi}'^*_0 \mathrm{e}^{-\mathrm{i}t}\}, \\ \hat{v}'_0 &= \frac{1}{2\mathrm{i}\,\mathrm{i}^{\frac{1}{2}}}(1 - \mathrm{e}^{-\mathrm{i}^{\frac{1}{2}} y}), \end{aligned}\right\} \tag{4.3}$$

where the asterisk denotes a complex conjugation.

To order κ we have from (2.24) for the residual flow in the Stokes' layer

$$\frac{\partial^4 \bar{v}'_1}{\partial y'^4} = (1-x)\,\overline{\sin t \frac{\partial}{\partial x}\frac{\partial^2 \psi'_0}{\partial y'^2}} + \overline{J'\left(\psi'_0, \frac{\partial^2 \psi'_0}{\partial y'^2}\right)} - \overline{\sin t \frac{\partial^2 \psi'_0}{\partial y'^2}}, \tag{4.4}$$

subject to

$$\left.\begin{aligned} &\bar{\psi}'_1 = 0, \frac{\partial \bar{\psi}'_1}{\partial y'} = 0 \quad (y' = 0), \\ &\frac{\partial \bar{\psi}'_1}{\partial y'} \rightarrow \text{constant finite value} \quad (y' \rightarrow \infty), \\ &\frac{\partial^2 \bar{\psi}'_1}{\partial y'^2} \rightarrow 0 \quad (y' \rightarrow \infty), \\ &\bar{\psi}'_1 = 0 \quad (x = 1). \end{aligned}\right\} \tag{4.5}$$

The third condition is the crucial one as it is not possible to require that the velocity vanishes outside the Stokes layer. Instead the relaxed condition of a finite residual velocity outside that layer is posed, as was the original idea of Schlichting (1932). Its validity has been discussed by Riley (1965) and Stuart (1966). It is of course this unphysical behaviour at infinity that has to be corrected for by the outer solution later on. Subject to the given conditions the solution of (4.4) reads:

$$\left.\begin{aligned}\overline{\psi}_1' &= (1-x)\{\phi(y')\,e^{-\frac{1}{2}\sqrt{2}y'}-\tfrac{3}{4}y'+\tfrac{7}{8}\sqrt{2}\},\\ \phi(y') &= -\tfrac{1}{4}\sqrt{2}\{\tfrac{1}{2}e^{-\frac{1}{2}\sqrt{2}y'}+\sin(\tfrac{1}{2}\sqrt{2}y')+3\cos(\tfrac{1}{2}\sqrt{2}y')\}.\end{aligned}\right\} \tag{4.6}$$

Note that (4.6) agrees with the boundary-layer part ($y\rightarrow 0$) of $\overline{\psi}' = \overline{\psi}/E$, where $\overline{\psi}$ is given in (3.9). The components of the residual velocity in the Stokes layer are given by

$$\left.\begin{aligned}\overline{u} &= \kappa(1-x)\{-e^{-ay}[\tfrac{1}{4}e^{-ay}+\sin(ay)+\tfrac{1}{2}\cos(ay)]+\tfrac{3}{4}\}+O(\kappa/a),\\ \overline{v} &= E\kappa\frac{\partial\overline{v}_1'}{\partial x}+O(\kappa/a),\end{aligned}\right\} \tag{4.7}$$

where $\overline{\psi}_1'$ is given in (4.6) and a in (A 2). These results agree with the boundary-layer parts of $\overline{u}$ and $\overline{v}$ in (3.10) and (3.11). Thus, for $y'\rightarrow\infty$ the residual velocity is $3(1-x)/4$. Being positive this means a net flow into the basin (flood surplus) at the top of the inner layer.

The constant residual velocity outside the Stokes layer means a non-zero stream function at the mid-basin axis if we were to extend the solution into that region. This has to be corrected for by properly choosing the mid-axis boundary conditions for the outer regime. We shall see that this choice also implies the vanishing of the residual velocity at the axis for basins which have effectively an infinite width, provided we include self-advection of residual vorticity.

In Appendix B the derivation of the equation for the $O(\kappa)$ residual stream function $\overline{\chi}_1$ of the outer regime is given. It obeys

$$\frac{1}{R_*}\frac{\partial^4\overline{\chi}_1}{\partial y^4} = J\left(\overline{\chi}_1,\frac{\partial^2\overline{\chi}_1}{\partial y^2}\right), \tag{4.8}$$

with the boundary conditions

$$\left.\begin{aligned}&\overline{\chi}_1 = 0, \quad \frac{\partial\overline{\chi}_1}{\partial y} = 0 \qquad \text{at } y=0,\\ &\overline{\chi}_1 = \alpha(1-x), \quad \frac{\partial^2\overline{\chi}_1}{\partial y^2} = 0 \quad \text{at } y=1.\end{aligned}\right\} \tag{4.9}$$

Here R_* is given by (1.3) and $\overline{\chi}_1$ and y are non-dimensional variables scaled as ψ and y in (2.25), i.e. with UB and B respectively. Furthermore

$$\left.\begin{aligned}\alpha &= \tfrac{3}{4}(1-E\alpha'),\\ \alpha' &= \tfrac{7}{6}\sqrt{2}+\tfrac{4}{3}\phi(E^{-1})\exp(-\tfrac{1}{2}\sqrt{2}E^{-1}),\end{aligned}\right\} \tag{4.10}$$

with ϕ defined in (4.6). Evidently, the third boundary condition in (4.9) causes the total $O(\kappa)$ solution $\overline{\psi}_1 = \overline{\chi}_1+E\overline{\psi}_1'$ to vanish at the basin axis $y=1$.

Obviously, if $R_*\ll 1$ in (4.8), self-advection can be neglected, whereas for $R_*\gg 1$ a new boundary layer must be expected since then R_*^{-1} is a small parameter multiplying the highest derivative in (4.8). We briefly consider the regime $R_*\ll 1$ first. Then we have as a first approximation to $\overline{\chi}_1$:

$$\frac{\partial^4\overline{\chi}_1}{\partial y^4} = 0, \tag{4.11}$$

subject to the boundary conditions (4.9). The solution reads

$$\overline{\chi}_1 = \alpha(1-x)\,(-\tfrac{1}{2}y^3+\tfrac{3}{2}y^2), \tag{4.12}$$

so that the total solution is given by

$$\overline{\psi}_1 = \overline{\chi}_1+E\overline{\psi}_1' = (1-x)\,[E\phi(y')\,\mathrm{e}^{-\frac{1}{2}\sqrt{2}y'}-\tfrac{1}{2}\alpha y^3+\tfrac{3}{2}\alpha y^2-\tfrac{3}{4}y+\tfrac{7}{8}\sqrt{2E}], \tag{4.13}$$

where $\phi(y')$ is given by (4.6). With α given by (4.10) we recover (3.9). In particular we have, neglecting order E and smaller, $\overline{u} = -3\kappa(1-x)/8$ for $y = 1$, implying a finite outward velocity at the basin axis, see also (3.12).

We now consider the regime $R^* = O(1)$ and larger. In this case we have to face the fully nonlinear equation (4.8). First we suppose that $\overline{\chi}_1$ has the now familiar form

$$\overline{\chi}_1 = (1-x)\,\Phi(y). \tag{4.14}$$

Substitution in (4.8) gives

$$\frac{1}{R_*}\frac{\mathrm{d}^4\Phi}{\mathrm{d}y^4} = \left[\frac{\mathrm{d}\Phi}{\mathrm{d}y}\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2}-\Phi\frac{\mathrm{d}^3\Phi}{\mathrm{d}y^3}\right], \tag{4.15}$$

subject to the boundary conditions

$$\left.\begin{aligned} \Phi = 0, \quad \frac{\mathrm{d}\Phi}{\mathrm{d}y} = 0 \quad (y = 0),\\ \Phi = \alpha, \quad \frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2} = 0 \quad (y = 1),\end{aligned}\right\} \tag{4.16}$$

where α is defined in (4.10). The third condition again assures a vanishing total stream function at $y = 1$. No exact solution of (4.15) satisfying (4.16) is known, whereas a perturbation approach makes no sense due to the strong nonlinear interaction. For strong nonlinear interactions, however, sometines an iterative series may do, as has been proposed for an analogous problem by Fettis (1956). The method has been proven to be useful in the present context (Stuart, 1966; Grotberg 1984). It consists of expanding first Φ in a power series of a formal expansion parameter, ϵ:

$$\Phi = \Phi_0+\epsilon\Phi_1+\epsilon^2\Phi_2+\dots. \tag{4.17}$$

Now Φ_0 is a 'first guess' testfunction that satisfies (4.15) and all but one of the boundary conditions in (4.16). Evidently

$$\Phi_0 = \text{constant} = \alpha \tag{4.18}$$

is an obvious choice satisfying all but the first conditions in (4.16). The first condition then reads:

$$\alpha+\epsilon\Phi_1(0)+\epsilon^2\Phi_2(0)+\dots = 0. \tag{4.19}$$

Substituting (4.17) and (4.18) into (4.15), collecting terms of equal powers in ϵ, gives an infinite series of linear differential equations for each of the iterates Φ_i $(i \geqslant 1)$:

$$\frac{1}{R_*}\frac{\mathrm{d}^4\Phi_1}{\mathrm{d}y^4} = \sum_{j=0}^{i-1}\left\{\frac{\mathrm{d}\Phi_j}{\mathrm{d}y}\frac{\mathrm{d}^2\Phi_{i-j}}{\mathrm{d}y^2}-\Phi_j\frac{\mathrm{d}^3\Phi_{i-j}}{\mathrm{d}y^3}\right\}, \tag{4.20}$$

subject to (4.19) and to

$$\Phi_i(1) = \frac{\mathrm{d}\Phi_i}{\mathrm{d}y}(0) = \frac{\mathrm{d}^2\Phi_i}{\mathrm{d}y^2}(1) = 0. \tag{4.21}$$

The iterative series (4.17) can be truncated after a finite number of terms. The truncated sum then approximates the solution of (4.15) subject to (4.16) after restoring $\epsilon = 1$. The convergence of this procedure has been demonstrated by Watson (1965). Here we shall truncate at the lowest possible order, i.e. $i = 1$. The solution of

$$\frac{d^4\Phi_1}{dy^4} = -\gamma\frac{d^3\Phi_1}{dy^3} \quad (\gamma = \alpha R_*), \tag{4.22}$$

subject to (4.21) and to the truncated form of (4.19), $\Phi_1(0) = -\alpha$, reads

$$\Phi_1(y) = A\,e^{-\gamma y} + By^2 + Cy + D, \tag{4.23}$$

where

$$\left.\begin{aligned} A &= \frac{\alpha}{[(1-\frac{1}{2}\gamma^2)\,e^{-\gamma}+\gamma-1]}, \\ B &= -\tfrac{1}{2}\gamma^2\,e^{-\gamma}A, \\ C &= \gamma A, \quad D = -\alpha - A. \end{aligned}\right\} \tag{4.24}$$

Thus the total $O(\kappa)$ solution reads

$$\overline{\psi}_1 = (1-x)\,[E\phi(y')\,e^{-\frac{1}{2}\sqrt{2}y'} - \tfrac{3}{4}y + \tfrac{7}{8}\sqrt{2}E + A\{e^{-\gamma y} - \tfrac{1}{2}\gamma^2\,e^{-\gamma}y^2 + \gamma y - 1\}], \tag{4.25}$$

with $\phi(y')$ given by (4.6), γ by (4.22) and A by (4.24). The components of the residual current read

$$\left.\begin{aligned} \overline{u} &= \kappa(1-x)\,\{-e^{-ay}[\tfrac{1}{4}\,e^{-ay} + \sin(ay) + \tfrac{1}{2}\cos(ay)] \\ &\qquad + \tfrac{3}{4} + \gamma A[e^{-\gamma y} + \gamma\,e^{-\gamma}y - 1]\} + O\left(\frac{\kappa}{a}\right), \\ \overline{v} &= \kappa\frac{\partial\overline{\psi}_1}{\partial x} + O\left(\frac{\kappa}{a}\right), \end{aligned}\right\} \tag{4.26}$$

where $\overline{\psi}_1$ is given in (4.25). At the mid-axis $y = 1$ the longitudinal component of the residual current is

$$\overline{u}(1) = \tfrac{3}{4}\kappa(1-x)\left\{1 + \frac{(1-E\alpha')\,\gamma(e^{-\gamma}+\gamma\,e^{-\gamma}-1)}{(1-\frac{1}{2}\gamma^2)\,e^{-\gamma}+\gamma-1}\right\} + O(\kappa E), \tag{4.27}$$

where use has been made of (4.10) and (4.24).

It is now interesting to consider two limits of (4.27). For $\gamma \to 0\,(R_* \ll 1)$ the contributions in (4.27) which are $O(\kappa\gamma E)$ can be neglected since they are $O(\kappa E)$. Next expanding the nominator and denominator in powers of γ we obtain

$$\overline{u}(1) = -\tfrac{3}{8}\kappa(1-x) + O(\kappa\gamma, \kappa E), \tag{4.28}$$

which is in agreement with (3.12). For $\gamma \to \infty\,(R_* \gg 1)$ the contributions in (4.27) which are $O(\kappa\gamma E)$ must be included since they can be $O(\kappa)$. We then have

$$\overline{u}(1) = -\tfrac{3}{4}\kappa(1-x)\left\{\frac{1-\alpha'\gamma E}{\gamma}\right\} + O\left(\frac{\kappa}{\gamma^2}, \kappa E\right). \tag{4.29}$$

It appears that the sign of the residual velocity will depend on the choices of γ and E. For γ small with respect to E^{-1} we will have an ebb surplus at the mid-axis. On the other hand, for $\gamma \geqslant E^{-1}$ a flood surplus occurs, since $\alpha' \approx 7\sqrt{2}/6$. However, the most important conclusion from (4.29) is that the residual current at the mid-axis is $O(1/\gamma)$ for $\gamma \to \infty$. This is due to the strong self-interaction mechanism, which forces the residual current to tend to zero in an outer boundary layer with a width $O(R_*^{-\frac{1}{2}})$.

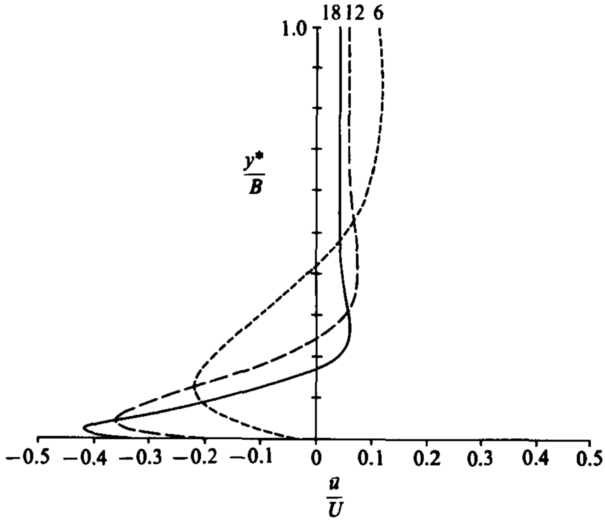

FIGURE 3. Lateral profiles of the non-dimensional longitudinal residual current velocity according to (4.26) for $\kappa = 0.7$ and for the values of E^{-1} shown on top. Flow at the mid-axis of the basin is negative.

5. Discussion and conclusions

We have shown that in a semi-enclosed tidal basin a rectified flow may develop due to vorticity-advection in lateral viscous boundary layers of thickness $(\nu/\sigma)^{\frac{1}{2}}$. The residual flow extends outside this oscillatory viscous wall layer and exhibits a boundary-layer character itself when the Reynolds number, R_*, becomes large. This double boundary-layer character is due to self-advection by the residual flow field. As $R_* = \kappa^2 E^{-2}$ this means that if we let the width of the basin increase relative to the Stokes-layer thickness $(\nu/\sigma)^{\frac{1}{2}}$, keeping κ = constant $\ll 1$, we will always encounter the regime $R_* \gg 1$. Then the outer boundary layer establishes itself such that residual velocities in the centre of the basin become very weak. The results obtained in the §§3 and 4 are valid for tidal basins having a width/length ratio $(2B/L)$ of 2. However they can easily be generalized to the case $B/L = O(1)$. The only differences are that then y represents the dimensional y-coordinate scaled by B, the Stokes number becomes $E = \delta/B$ and that the results for $\bar{v}$ in (3.11), (4.7), (4.26) and (A 8) should be multiplied by B/L.

Of course, the regime that will prevail in reality depends on the parameters that are representative for a typical tidal basin. As for this we assume the following characteristic values to apply: tidal velocity amplitude $U = 1\ \mathrm{ms^{-1}}$, basin length $L = 10$ km, basin width $B = 5$ km, tidal frequency $(M_2)\,\sigma = 1.4\times 10^{-4}\ \mathrm{s^{-1}}$, whereas for the most uncertain parameter, the lateral turbulent viscosity coefficient ν we assume a range 10–100 $\mathrm{m^2s^{-1}}$. Then $\kappa = 0.7$, for which our analysis is marginally valid; $\delta = (\nu/\sigma)^{\frac{1}{2}}$ is between 270 and 850 m, hence $E^{-1} = B/\delta$ is between 6 and 19; finally R_* varies between 18 and 180. These values show that the Stokes layers are always small compared to the basin width and, more important, that in reality $R_* \gg 1$ is much more likely than the regime $R_* \ll 1$ which has been discussed before by Yasuda (1980). In figure 3 we show the lateral profile of the longitudinal component of the residual velocity according to (4.26) for a range of values of R_*, for fixed $\kappa = 0.7$; i.e. we vary E^{-1} between 6 and 18. As can be seen the boundary layer character of the residual current increases with increasing R_*. Notably the structure

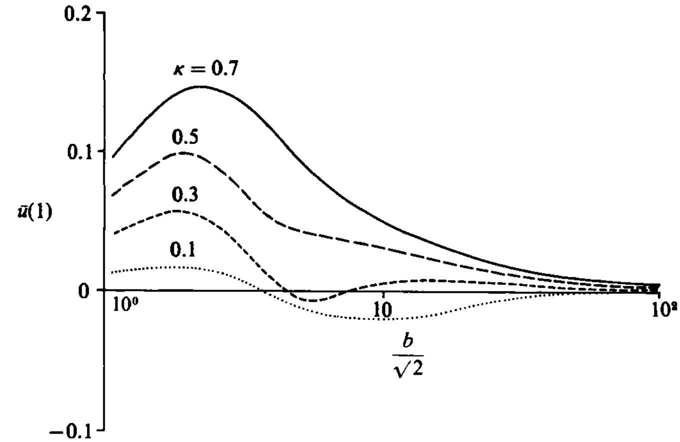

FIGURE 4. Longitudinal residual velocity $\bar{u}(1)/(1-x)$ according to (4.27) as a function of $^{10}\log a$, where $a = (\sqrt{2}E)^{-1}$, and for the values of κ shown on top. The flow in the central region of the basin can be negative as well as positive.

of it is different from that in figure 1 where the double boundary-layer character has not been taken into account. In the latter case the circulation has two cells with water flowing into the basin (flood surplus) in the middle of the half width whereas there is ebb surplus near the sidewalls and at the mid-basin axis. In the case of a large R_* however, figure 3 shows that the structure is more simple having an ebb surplus in the boundary layer and a flood surplus being spread out over the cross-section, leading to weak residual currents in the centre of the basin and to strong currents near the wall. Actually, as $\bar{u}$ is scaled by κU (with $U = 1\ \mathrm{ms}^{-1}$) the residual speed in the boundary layer can be quite substantial, several decimetres per second.

In figure 4 the dependence of the longitudinal residual velocity at the mid-axis of the basin, given by (4.27), on the Stokes number is shown for different values of the Strouhal number. This is supplied in order to show that $\bar{u}(1)$ tends to zero for $R_* \rightarrow \infty$. Furthermore, it appears that both an ebb surplus and flood surplus can occur, as has been discussed in §4. However, for realistic situations ($\kappa \approx 0.7$) a flood surplus in the central region is more likely to be the case.

This single-cell structure accords with qualitative vorticity arguments given by Yasuda (1980) and Zimmerman (1981). It is also the structure that arises when bottom friction dominates vorticity dissipation relative to lateral diffusion (Yasuda 1980). However if the latter is not the case our results show that self-advection of the residual current is a necessary ingredient for the single-cell structure to occur.

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Appendix A. Residual current solution for $\kappa \ll 1, R_* = \kappa^2 E^{-2} \ll 1$

Consider equation (3.8), being the $O(\kappa)$ residual part of the vorticity equation (2.26) for $R_* \ll 1$. The function ψ_0 on the right-hand side is given in (3.3)–(3.4). The solution $\overline{\psi}_1$, subject to the boundary conditions (2.27), except the fourth one which should be replaced by (3.6), can be obtained by integration. It reads

$$\overline{\psi}_1(x,y) = (1-x)\,\overline{\phi}_1(\xi), \tag{A 1}$$

where
$$\xi = \alpha(1-y), \quad a = (\surd 2E)^{-1}, \tag{A 2}$$

and

$$\begin{aligned}\overline{\phi}_1(\xi) = -\frac{1}{\beta a}\Big\{&\tfrac{1}{4}[\sinh(2\xi)-\sin(2\xi)]\\&+\left[2(\sinh a\,\sin a+\cosh a\,\cos a)+\frac{3}{\alpha}\cosh a\,\sin a\right]\sinh\xi\,\cos\xi\\&+\left[2(\sinh a\,\sin a-\cosh a\,\cos a)-\frac{3}{\alpha}\sinh a\,\cos a\right]\cosh\xi\,\sin\xi\\&+\frac{\xi}{a}[(\sinh a\,\cos a+\cosh a\,\sin a)\sinh\xi\,\sin\xi\\&+(\sinh a\,\cos a-\cosh a\,\sin a)\cosh\xi\,\cos\xi]\Big\}+C_1\xi^3-C_2\xi.\end{aligned} \tag{A 3}$$

Furthermore,

$$\begin{aligned}\beta = 4\Big\{&\cos^2 a+\cosh^2 a-\frac{1}{\alpha}[\sinh a\,\cosh a+\sin a\,\cos a]\\&+\frac{1}{2a^2}[\sin^2 a+\sinh^2 a]\Big\},\end{aligned} \tag{A 4}$$

$$C_1 = \frac{1}{\beta a^4}\{\tfrac{3}{4}\alpha[\cosh(2a)-\cos(2a)]-\tfrac{11}{8}[\sinh(2a)-\sin(2a)]\}, \tag{A 5}$$

$$C_2 = \frac{1}{\beta a^2}\{\tfrac{3}{4}\alpha[\cosh(2a)-\cos(2a)]-\tfrac{25}{8}[\sinh(2a)-\sin(2a)]\}. \tag{A6}$$

The components $(\overline{u}, v)$ of the residual current become

$$\overline{u} = -\kappa\frac{\partial\overline{\psi}_1}{\partial y}+\ldots = \kappa\alpha(1-x)\frac{\mathrm{d}\overline{\phi}_1}{\mathrm{d}\xi}+\ldots, \tag{A 7}$$

$$\overline{v} = \kappa\frac{\partial\overline{\psi}_1}{\partial x}+\ldots = -\kappa\overline{\phi}_1(\xi)+\ldots, \tag{A 8}$$

which can easily be calculated from (A 3).

Appendix B. Residual current dynamics for $\kappa \ll 1, R'_* = \kappa^2E^{-2} = O(1)$ and larger

Consider the vorticity equation (2.26). If $E \ll 1$ it can be analysed by boundary-layer methods. The solution is written as

$$\psi = \chi(x,y,t)+E\psi'(x,y',t). \tag{B 1}$$

Here $y' = y/E$ is a boundary-layer coordinate, χ is an outer solution obeying (2.26) and ψ' an inner solution obeying (2.24). The boundary conditions are

$$\left.\begin{array}{lll} \chi = 0, & \psi' = 0 & \text{at } y = 0, x = 1, \\ \dfrac{\partial \chi}{\partial y} = 0, & \dfrac{\partial \psi'}{\partial y'} = (1-x)\sin t & \text{at } y = 0, \\ \dfrac{\partial^2 \chi}{\partial y^2} = 0, & \dfrac{\partial^2 \psi'}{\partial y'^2} = 0 & \text{at } y = 1, \\ \chi + \dfrac{1}{b}\psi' = 0 & & \text{at } y = 1, \end{array}\right\} \tag{B 2}$$

which can be derived from (2.24)–(2.27) and (B 1).

In this case $E^2 = \kappa^2/R_*$, which clearly is small. This suggests expanding χ and ψ' in power series of the small parameter κ:

$$\chi = \sum_{n=0}^{\infty} \kappa^n \chi_{0n}, \quad \psi' = \sum_{n=0}^{\infty} \kappa^n \psi'_n. \tag{B 3}$$

First consider the zeroth-order system. The inner solution ψ'_0 is already obtained in §4, see (4.3). From application of (B 2) it then follows $\chi_0 = 0$ for the outer solution.

From the last condition in (B 2) it can be seen that ψ'_0 forces an $O(\kappa/\sqrt{R_*})$ outer solution. In $O(\kappa)$ a residual current is generated which extends into the outer layer. Consequently the functions χ_n and $\psi'_n\,(n \geqslant 1)$ in (B 3) will consist of residual parts $\overline{\chi}_n, \overline{\psi}'_n$ and oscillating parts χ^t_n, ψ'^t_n. The residual current dynamics in lowest order follows from consideration of (2.26) for ψ in $O(\kappa^3)$ and (2.24) for ψ' in $O(\kappa)$. Using $\chi_0 = 0$ we find

$$\frac{1}{R_*}\frac{\partial^4 \overline{\chi}_1}{\partial y^4} = \overline{(1-x)\sin t\frac{\partial}{\partial x}\frac{\partial^2 \chi_2}{\partial y^2}} + \overline{J\left(\chi_1, \frac{\partial^2 \chi_1}{\partial y^2}\right)} - \overline{\sin t\frac{\partial^2 \chi_2}{\partial y^2}}, \tag{B 4}$$

while $\overline{\psi}'_1$ obeys (4.4). The solution of the latter equation, subject to the boundary conditions (B 2), is given in (4.6).

In (B 4) we must specify the parts $\chi^{t,0}_2$ of χ^t_2 which oscillates with the basic tidal frequency, as well as the oscillating part χ^t_1 of χ_1. This can be done by studying the $O(\kappa)$ and $O(\kappa^2)$ dynamics in the outer layer. From (4.26) for χ and (4.3) we obtain in $O(\kappa)$

$$\frac{\partial}{\partial t}\frac{\partial^2 \chi_1}{\partial y^2} = 0. \tag{B 5}$$

Consequently χ^t_1 is only driven by the zeroth-order inner solution ψ'_0, which according to (4.3) is proportional to $\sin t$. Consequently, from (B 5),

$$\chi^t_1 \sim \sin t, \frac{\partial^2 \chi^t_1}{\partial y^2} = 0. \tag{B 6}$$

The $O(\kappa^2)$ dynamics of (4.26) for χ reads

$$\frac{\partial}{\partial t}\frac{\partial^2 \chi_2}{\partial y^2} + (1-x)\sin t\frac{\partial}{\partial x}\frac{\partial^2 \chi_1}{\partial y^2} - \sin t\frac{\partial^2 \chi_1}{\partial y^2} = 0. \tag{B 7}$$

As can be seen χ^t_2 is driven by χ_1, but also by the $O(\kappa)$ inner-layer solution ψ'^t_1 due to the last condition of (B 2). However, since we are only interested in $\chi^{t,0}_2$, which oscillates with the basic tidal frequency, the forcing due to ψ'^t_1 need not be considered as it oscillates with twice the tidal frequency. Then, from (B 7), only $\overline{\chi}_1$ contributes to $\chi_2^{t,0}$ and we conclude

$$\chi_2^{t,0} \sim \cos t. \tag{B 8}$$

Substitution of (B 6) and (B 8) in (B 4) gives (4.8) with the boundary conditions (4.9).

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